Evaluation of Harmonic Coupling Weights in Nonlinear Periodicity Preservation Systems

Charles Baylis, II, *Member, IEEE*, and Robert J. Marks, II, *Fellow, IEEE*

***Abstract*—When stimulated by a periodic stimulus, every component of a possibly nonlinear periodicity preservation system oscillates periodically with the same period as the stimulus. Changes in the harmonic structure of the the stimulus couples each input harmonic to, in general, every response harmonic. Knowledge of harmonic coupling weights (HCWs) allows model free characterization of effects of all small stimulus perturbation. This paper develops foundational methodology for experimentally measuring the HCWs. The on-frequency method uses small tickle tones at harmonic frequencies to do this. The off-frequency method places tickle tones adjacent to harmonic frequencies and is applicable to PP systems that are locally frequency invariant. Memoryless nonlinear PP systems are an example where the approach works exactly. The off-frequency method also requires fewer experiments. In some cases, the off-frequency method can be extended to measure numerous HCWs in a single experiment. Some system types such as the memoryless nonlinearity, require fewer experiments than general PP systems. Methods to experimentally measure the Hessian are also presented.**

***Index Terms*—Affine approximation, efficient, harmonic coupling, harmonics, nonlinear systems, power amplifier.**

I. Introduction

AFFINE approximations to nonlinearities have found recent popularity as amplifiers are are being driven into nonlinear operation to achieve greater efficiency. Of specific interest in coupling among harmonics imposed by the perturbations around nonlinearities [1]–[24]. In previous papers [3], we have outlined the affine approximation and the relationships among time, frequency, and mixed time frequency characterizations of affine approximations of nonlinear *periodicity preservation* (PP) systems. The methodology, developed for single port networks, was shown to be generalizable to multiport systems. In this paper, we examine experimental procedures to measure harmonic coupling weights in a PP nonlinear system around an operating point. These weights are conventional phasors that contain the relationship between stimulus and response amplitudes albeit at generally different harmonics.

Manuscript received December 29, 2011; revised April 01, 2012; accepted April 21, 2012. Date of current version November 21, 2012. An earlier abbreviated version of the multiplexing off-frequency tickle tone section was been presented by the authors [1]. A tutorial introduction to this material was presented by the authors in IEEE *Microwave Magazine* [2]. This work was supported by a grant from the U.S. Naval Research Laboratory. This paper was recommended by Associate Editor I. Belykh.

The authors are with the Wireless and Microwave Circuits and Systems Program, Department of Electrical and Computer Engineering Department, Baylor University, Waco, TX 76798-7356 USA (e-mail: Robert_Marks@baylor.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSI.2012.2206441

In this paper we show the following.

1. The *harmonic coupling weights* (HCWs), equal to the Fourier series Jacobian of the nonlinear transformation about the operating point, can be determined from the response of the PP system to small amplitude complex sinusoid *tickle tones*.
2. Real sine and cosine tickle tones can also be used to experimentally to find the HCWs by superimposing the results of two experiments. The procedure requires calculating the perturbation differences in the stimulus and response.
3. When the PP system is a memoryless nonlinearity, a single set of coupling coefficients is required. Other HCWs are shifted versions. The response perturbation is determined by a discrete convolution of Fourier coefficients [3].
4. When the PP system is locally frequency invariant and there is no cross harmonic interference, an off-frequency experiment allows placement of the tickle tones adjacent rather than at the harmonic frequency. The amplitudes can be measured directly rather than as the difference between the original and perturbed response at a harmonic frequency. A single off-frequency experiment replaces the two required on-frequency experiments (sine and cosine).
5. More generally, tickle tones can be frequency multiplexed. Under local frequency invariance constraints and no cross coupling, a generalization of the off-frequency method allows experimental measurement of numerous HCWs in a single experiment.
6. All memoryless nonlinearities are globally frequency invariant and display no cross harmonic coupling or interference.
7. The tickle tone approach can be extended to measure higher order derivative measures such as the second order Hessian.

II. Background

Let $\mathcal{Z}$ denote a nonlinear operator and

$$v(t) = \mathcal{Z}\{i(t)\}. \tag{1}$$

The signal $i(t)$ is the stimulus and $v(t)$ the response. The affine approximation for a small stimulus perturbation, $\Delta i(t)$, is [2], [3]

$$v_\Delta(t) := \mathcal{Z}\{i(t) + \Delta i(t)\} \approx v(t) + \Delta v(t) \tag{2}$$

where

$$\Delta v(t) = \int_{-\infty}^{\infty} \frac{\partial v(t)}{\partial i(\tau)} \Delta i(\tau) d\tau \tag{3}$$

There are versions of (3) that use the Fourier transforms of the stimulus and response [3]. For example,

$$\Delta v(t) = \int_{-\infty}^{\infty} \frac{\partial v(t)}{\partial I(\nu)} \Delta I(\nu)\, d\nu \tag{4}$$

where the Fourier transform of $i(\tau)$ is

$$I(\nu) = \int_{-\infty}^{\infty} i(\tau)\, e^{-j2\pi\nu\tau} d\tau \tag{5}$$

and

$$\frac{\partial v(t)}{\partial I(\nu)} = \int_{-\infty}^{\infty} \frac{\partial v(t)}{\partial i(\tau)} e^{j2\pi\nu\tau} d\tau \tag{6}$$

The *periodicity preservation* (PP) class of nonlinear systems [3] has the property that periodic stimulation will result in a periodic response with the same period, T and fundamental frequency $f = 1/T$. Temporally, the PP system is represented by the operator $\mathcal{Z}$ with the constraint that the stimulus and response in (1) are zero outside of a period, say the interval $0 \leq t < T$. Each defines a single period of a periodic function. For example a periodic signal $\hat{v}(t)$ can be defined by replication of a single period.

$$\hat{v}(t) = \sum_{p=-\infty}^{\infty} v(t - nT). \tag{7}$$

Since $\Delta i(t)$ is nonzero only over $0 \leq t < T$, (3) becomes

$$\Delta v(t) = \int_{0}^{T} \frac{\partial v(t)}{\partial i(\tau)} \Delta i(\tau) d\tau \tag{8}$$

Let the vector i_m denote the Fourier coefficients of the periodic signal $\hat{l}(\tau) = \sum_{p=-\infty}^{\infty} i(\tau - nT)$. That is

$$i_m = \frac{1}{T} \int_{0}^{T} i(\tau) e^{-j2\pi m f \tau} d\tau \tag{9}$$

and the Fourier series [26] is

$$i(\tau) = \sum_{m=-\infty}^{\infty} i_m e^{j2\pi m f \tau} \Pi_T(\tau) \tag{10}$$

where $\Pi_T(t)$ is one for $0 \leq t < T$ and is otherwise zero. From (9), the Fourier coefficients are seen to be samples of the Fourier transform of a single period of $\hat{l}(t)$ That is,

$$i_m = fI(mf). \tag{11}$$

where, since $i(\tau)$ is identically zero outside of the interval $0 \leq t < T$, (5) can be written as

$$I(\nu) = \int_{0}^{T} i(\tau)\, e^{-j2\pi\nu\tau} d\tau,$$

Likewise, let v_n denote the Fourier coefficients of the periodic signal $\hat{v}(t)$. Then (8), expressed in terms of Fourier series coefficients, is the harmonic cross coupling expression [3]

$$\Delta v_n = \sum_{m=-\infty}^{\infty} \frac{\partial v_n}{\partial i_m} \Delta i_m \tag{12}$$

where Δv_n and Δi_m are the Fourier series coefficients of $\Delta v(t)$. The *harmonic coupling weights* (HCWs), $\partial v_n / \partial i_m$ measure the coupling strength between the mth stimulus harmonic and the nth response harmonic. Only a finite number of experiments can be performed and a finite number of HCWs experimentally determined. Therefore, in lieu of (12), we approximate the perturbation of Fourier series coefficients

$$\Delta v_n \simeq \sum_{m=-M}^{M} \frac{\partial v_n}{\partial i_m} \Delta i_m\, ; \quad |n| \leq N. \tag{13}$$

Thus, with knowledge of the HCWs, the Fourier series coefficients of a response perturbation can be estimated for a small albeit arbitrary stimulus perturbation. In the sections to follow, we illustrate a foundational methodology by which the harmonic coupling weights of a model free PP system can be determined experimentally.

III. Measuring the Harmonic Coupling Weights the On-Frequency Method

Consider the PP circuit in Fig. 1(a). The stimulus, with Fourier coefficients i_m, give rise to a periodic response with coefficients v_n. In Fig. 1(b), the kth stimulus harmonic is perturbed from i_k to one with Fourier coefficients $i_k + \Delta i_k$.

$$\Delta_i(t) = \varepsilon\, e^{j2\pi k f}. \tag{14}$$

where ε is a small amplitude. The response to this small *tickle tone* is a perturbation of v_n to an affine approximation of $v_n + \Delta v_n$. The single frequency tickle tone in (14) is complex and not realizable, but serves as a useful pedagogical introduction to experimental determination of HCWs. The segue into the use of real tickle tones follows smoothly.

The tickle tone in (14) has a single Fourier series coefficient, i.e.,

$$\Delta i_m = \varepsilon\, \delta[m - k] \tag{15}$$

where the Kronecker delta, $\delta[\ell]$, is one for $\ell = 0$ and is otherwise zero. Substituting into (12), the response perturbation is

$$\Delta v_n = \varepsilon \frac{\partial v_n}{\partial i_k}. \tag{16}$$

The output's affine approximation follows as $v_n + \varepsilon \partial v_n / \partial i_k$. The HCWs, $\partial v_n / \partial i_k$ can therefore be measured for all n by subtracting the Fourier series of the perturbed response from the Fourier coefficients at the operating point, v_n, and dividing by ε. All coupling weights from the kth input harmonic can be thereby determined. The process is repeated for every stimulus harmonic corresponding to different values of k and all HCWs are found. The result can then be used to estimate the response to any small stimulus perturbation around the operating point.

Since the tickle tone is applied directly on the harmonic, this approach and those similar are appropriately dubbed *on-frequency methods*.

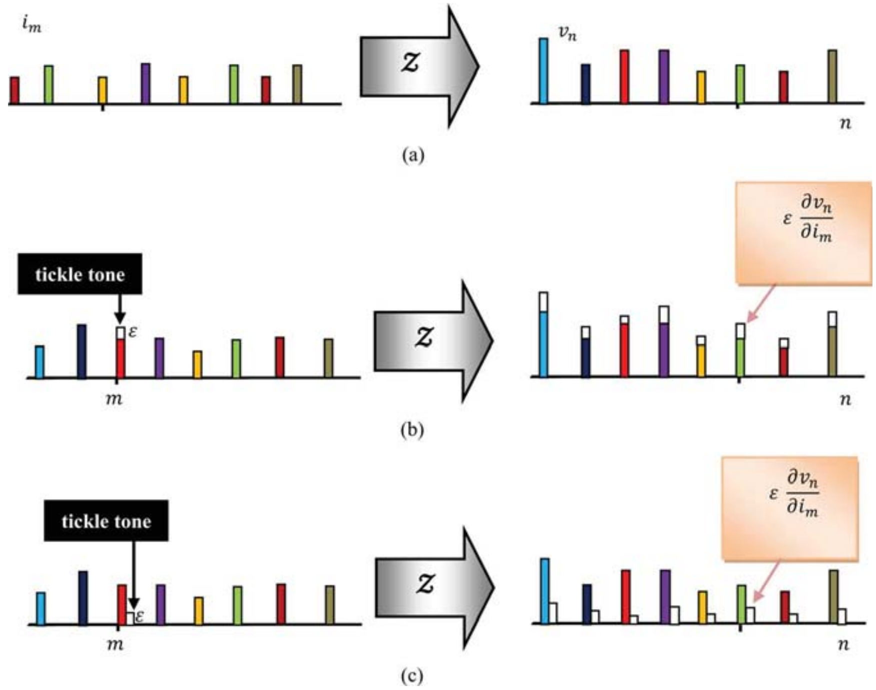

Fig. 1. Illustration of the two methodologies for measuring the harmonic coupling weights. (a) The operating point mapping input Fourier coefficients, i_m to the output, v_n. (b) For the on-frequency approach, a tickle tone, $\varepsilon\ e^{j2\pi mft}$, is added onto the mth harmonic. The increments at the nth output harmonic are approximations of $\varepsilon(\partial v_n/\partial i_m)$. (c) For the off-frequency approach, a tickle tone $\varepsilon\ e^{j2\pi(mf+\Delta f)t}$ is added a frequency interval $\Delta\nu$ from the mth harmonic. If the frequency invariant property holds, the terms a frequency interval Δf from the nth output harmonic are now approximations of $\varepsilon(\partial v_n/\partial i_m)$.

A. Experiments Using Real Signals

The analysis in the previous section requires stimulus of the PP system with complex sinusoids in (14). Using real signals, the HCWs $\partial v_n/\partial i_k$ and $\partial v_n/\partial i_{-k}$ can be found from two experiments using two real tickle tone perturbations.

The first perturbation of the input's kth harmonic uses the stimulus

$$\Delta_k^c i(t) = 2\varepsilon\cos(2\pi kft) \tag{17}$$

and the second uses the quadrature signal

$$\Delta_k^s i(t) = 2\varepsilon\sin\left(2\pi kft\right). \tag{18}$$

The Fourier series coefficients for these signals are, respectively,

$$\begin{aligned}\Delta_k^c i_m &= \varepsilon(\delta[m-k]+\delta[m+k])\\ &\text{and}\\ \Delta_k^s i_m &= -j\varepsilon(\delta[m-k]-\delta[m+k]).\end{aligned}$$

The affine approximations of the system response from (12) follow as

$$\Delta_k^c v_n = \varepsilon\left(\frac{\partial v_n}{\partial i_k}+\frac{\partial v_n}{\partial i_{-k}}\right), \tag{19}$$

and

$$\Delta_k^s v_n = -j\varepsilon\left(\frac{\partial v_n}{\partial i_k}-\frac{\partial v_n}{\partial i_{-k}}\right). \tag{20}$$

These two values are measured experimentally. From the two measurements, we evaluate the HCWs

$$\frac{\partial v_n}{\partial i_k} = \frac{1}{2\varepsilon}\left(\Delta_k^c v_n + j\Delta_k^s v_n\right) \tag{21}$$

and

$$\frac{\partial v_n}{\partial i_{-k}} = \frac{1}{2\varepsilon}\left(\Delta_k^c v_n - j\Delta_k^s v_n\right). \tag{22}$$

Measurements are repeated for all desired input harmonics corresponding to any values of k and positive n.

B. Harmonic Coupling for Harmonics With Negative Index

For negative n, note that, for real signals, $i_{-k} = i_k^*$ and

$$\frac{\partial v_n}{\partial i_{-k}} = \frac{\partial v_n}{\partial i_k^*}$$

Likewise, for real $v(t)$, $v_{-n} = v_n^*$ so that

$$\frac{\partial v_{-n}}{\partial i_k} = \frac{\partial v_n^*}{\partial i_k} = \left(\frac{\partial v_n}{\partial i_k^*}\right)^* = \left(\frac{\partial v_n}{\partial i_{-k}}\right)^*.$$

Therefore, negative indices on the output Fourier coefficients can be evaluated by conjugating the HCWs measured in (21) and (22) [3].[1]

[1]As we have previously noted [3], the use of Wirtinger calculus expressions as used, for example, in X parameters, is not necessitated.

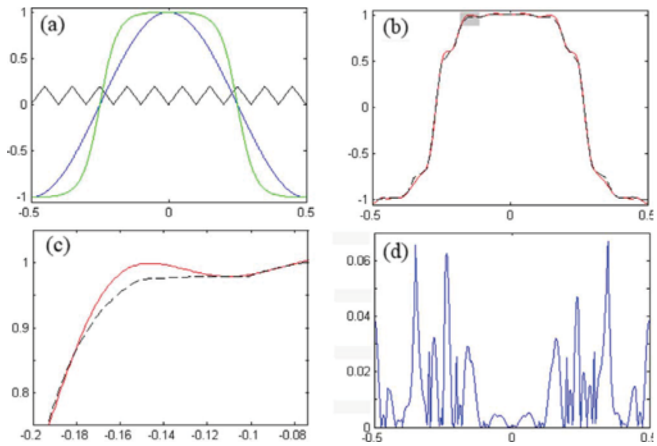

Fig. 2. The on-frequency appoximation using, as an input operating point, a unit amplitude cosine with period $T = 1$. The PP system is a memoryless nonlinear sigmoid given by (27). A value of $\vartheta = 0.4$ is used. The effect, as shown in (a), is to flatten the cosine. From this information, the harmonic coupling weights are estimated using the on-frequency method with $\varepsilon = 0.01$. Twenty positive frequency harmonics are used. The matrix of harmonic coupling weights is therefore 41×41 (Thus, with reference to (13), $N = M = 20$) A triangular wave perturbation is applied as shown in (a). The preturbation ranges between a minimum of zero and maximum of 0.2. The true PP system output for the perturbed cosinusoid is shown in (b) using a dashed line. The output computed using the harmonic coupling affine approximation is shown by the solid line. Detail of the small shaded box in (b) is shown in (c). The maximum absolute error [the magnitude of the difference between the two curves in (b)] is shown in (d). The overall RMS error using the affine approximation is 0.0189.

C. Memoryless Nonlinearities

A memoryless nonlinearity is a special case of the PP system. We now show that experiments *for a single sine and cosine stimulus suffice to completely specify* all HCWs for a memoryless nonlinearity. Let

$$v(t) = \mathcal{Z}\{i(t)\} = g(i(t)) \tag{23}$$

where $g(i)$ is a memoryless nonlinearity. Using the expression for the Fourier series coefficient of the response $v(t)$ results in

$$v_n = \frac{1}{T}\int_0^T g\left(\sum_{k=-\infty}^{\infty} i_k e^{j2\pi k\nu t}\right) e^{-j2\pi n\nu t} dt$$

and differentiating gives

$$\frac{\partial v_n}{\partial i_m} = h_{n-m} \tag{24}$$

where

$$h_n = \frac{1}{T}\int_{-\infty}^{\infty} \dot{g}(i(t)) e^{-j2\pi n\nu t} dt. \tag{25}$$

Then the HCW series in (12) becomes a discrete convolution.

$$\Delta v_n = \sum_{m=-\infty}^{\infty} h_{n-m}\Delta i_m = h_n * \Delta i_n \tag{26}$$

where the asterisk denotes convolution. This is good news. Since the HCWs, $\partial v_n/\partial i_m$, are a function of $n - m$, the expression in (24) reveals *the HCW matrix for memoryless nonlinearities is Toeplitz and knowledge of one row suffices to define the entire matrix*. Thus, only two single on-frequency measurements, one for sine and one for cosine, are required to define all HCWs from h_n.

1) Example 1: Sigmoid Nonlinearity: Let the PP system be characterized by a sigmoid parameterized by a positive number ϑ.

$$g(i) = \frac{1 + e^{-\vartheta}}{1 + e^{-\vartheta i}}. \tag{27}$$

For a cosine input, as illustrated in Fig. 2(a), the effect is a flattening of the cosine curve. The cosine, which is the input operating point, is perturbed by the small triangular wave shown in Fig. 2(a). The true perturbed output (continuous line) is favorably compared to the output estimated using HCW's (dashed line) in Fig. 2(b). Detail of the small shaded box in Fig. 2(b) is shown in Fig. 2(c). The absolute error between the actual output and the approximation is in Fig. 2(d). Further details are given in the caption of Fig. 2.

2) Example 2: Polynomial Nonlinearities: Polynomial nonlinearities, a special case of the memoryless nonlinearity in (23), have HCWs that can be characterized analytically. Define the Pth order polynomial

$$g(i) = \sum_{p=0}^{P} \alpha_p i^p \tag{28}$$

where the α_p's are real coefficients. Thus,

$$v(t) = g(i(t)) = \sum_{p=0}^{P} \alpha_p i^p(t)$$

and, for small input perturbations,

$$\begin{aligned} g\left(i\left(t\right) + \Delta i\left(t\right)\right) &:= \sum_{p=0}^{P} \alpha_p (i\left(t\right) + \Delta i\left(t\right))^p \\ &\approx \sum_{p=0}^{P} \alpha_p \left(i^p\left(t\right) + p i^{p-1}\left(t\right)\Delta i\left(t\right)\right) \\ &= v\left(t\right) + \Delta v\left(t\right) \end{aligned} \tag{29}$$

where we have discarded $(\Delta i(t))^p$ terms when $p \geq 2$. The approximation to the output perturbation follows as

$$\Delta v\left(t\right) = \sum_{p=0}^{P} \alpha_p p i^{p-1}\left(t\right)\Delta i\left(t\right). \tag{30}$$

In terms of Fourier series coefficients, an equivalent statement is

$$\Delta v_n = \sum_{p=0}^{P} \alpha_p p i_n^{(p-1)_*} * \Delta i_n \tag{31}$$

where the asterisk denotes discrete convolution and i_m^{p*} denotes the p fold autoconvolution of the Fourier series coefficients of $i(t)$. This is a special case of the convolution in (26) with

$$h_n = \sum_{p=0}^{P} \alpha_p p i_n^{(p-1)^*}. \tag{32}$$

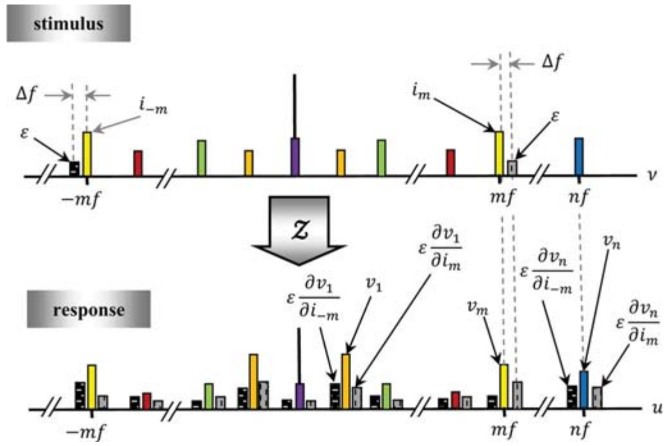

Fig. 3. Illustration of the off-frequency method of determining harmonic coupling weights for a PP system. On top is the spectrum of the stimulus, $i(\tau)$ and the spectrum of the response, $v(t)$, is on the bottom. Both signals are periodic with period $T = 1/f$ and therefore have their harmonics at integer values of f. A tickle tone, $\Delta i(\tau) = 2\varepsilon\cos(2\pi(mf + \Delta f)\tau)$, is added and appears in the stimulus spectrum with weight ε at the two frequencies $\pm(mf + \Delta f)$. If the PP system, $\mathcal{Z}$, obeys the harmonic noninterference and frequency invariant constraints, the harmonic coupling weights, $\varepsilon(\partial v_n/\partial i_m)$ and $\varepsilon(\partial v_n/\partial i_{-m})$ can be read at the output at frequencies $nf + \Delta f$ and $nf - \Delta f$ for all values of n.

This example will be continued in the description of the off-frequency method of finding the affine approximation for purposes of illustrating frequency invariance.

IV. The Off-Frequency Method

Finding the HCWs by placing the tickle tone a bit off-frequency is dubbed, appropriately, the *off-frequency method*. Then, as shown in Fig. 3(c), the responses to determine the HCWs are likewise displaced in frequency. When a real cosine tickle tone is used, there is an additional advantage to the off-frequency approach. A single experiment suffices to determine all of the parameters for a specified input harmonic whereas the on-frequency method requires two.

The off-frequency method works with locally frequency-invariant PP systems. If a system is *frequency-invariant* [27]–[31], then for all real values of α,

$$\frac{\partial V(u)}{\partial I(\nu)} = \frac{\partial V(u-\alpha)}{\partial I(\nu-\alpha)}. \tag{33}$$

A system is locally frequency-invariant if (33) is approximately true for small frequency shifts in the neighborhood about $\alpha = 0$.

Let a tickle tone be applied at β Hertz.

$$\Delta i(\tau) = \varepsilon e^{j2\pi\beta\tau}\Pi_T(\tau).$$

Then (3) becomes

$$\Delta v(t) = \varepsilon \int_0^T \frac{\partial v(t)}{\partial i(\tau)} e^{j2\pi\beta\tau} d\tau \tag{34}$$

From (6), this can be written as

$$\Delta v(t) = \varepsilon \frac{\partial v(t)}{\partial I(\beta)} \tag{35}$$

Fourier transforming both sides gives [3]

$$\Delta V(u) = \varepsilon \frac{\partial V(u)}{\partial I(\beta)} \tag{36}$$

Since, from (11), sampling the frequency domain gives Fourier series coefficients, we have, for $u = nf$ and $\beta = mf$.

$$\Delta V(nf) = \varepsilon \frac{\partial V(nf)}{\partial I(mf)} = \varepsilon \frac{\partial v_n}{\partial i_m} \tag{37}$$

This is an alternate description of the on-frequency method.

If the tickle tone, however, is moved to frequency $\beta = mf + \Delta f$, the tickle tone becomes

$$\Delta i(\tau) = \varepsilon e^{j2\pi(mf+\Delta f)\tau}\Pi_T(\tau) \tag{38}$$

and attention is focused on the response frequencies $u = nf + \Delta f$. Then the off-frequency method assumes

$$\begin{aligned}\Delta V(nf + \Delta f) &= \varepsilon \frac{\partial V(nf + \Delta f)}{\partial I(mf + \Delta f)} \approx \varepsilon \frac{\partial V(nf)}{\partial I(mf)} \\ &= \varepsilon \frac{\partial v_n}{\partial i_m}\end{aligned} \tag{39}$$

This is true when the PP system is locally frequency-invariant.

Alternately, a tickle tone can be placed slightly below a harmonic frequency. If

$$\Delta i(\tau) = \varepsilon e^{j2\pi(kf-\Delta f)\tau}\Pi_T(\tau)$$

then, following the same reasoning, we measure the HCWs at frequencies translated Δf above each harmonic.

$$\begin{aligned}\Delta V(nf - \Delta f) &= \varepsilon \frac{\partial V(nf - \Delta f)}{\partial I(kf - \Delta f)} \approx \varepsilon \frac{\partial V(nf)}{\partial I(kf)} \\ &= \varepsilon \frac{\partial v_n}{\partial i_k}\end{aligned} \tag{40}$$

Likewise, the HCWs corresponding to input Fourier series coefficients with negative indices can be measured at intervals Δf below each harmonic frequency. Specifically, if $k = -m$, then

$$\begin{aligned}\Delta V(nf - \Delta f) &= \varepsilon \frac{\partial V(nf - \Delta f)}{\partial I(-mf - \Delta f)} \approx \varepsilon \frac{\partial V(nf)}{\partial I(-mf)} \\ &= \varepsilon \frac{\partial v_n}{\partial i_{-m}}\end{aligned} \tag{41}$$

Implementing the off-frequency technique is illustrated in Fig. 3.

Theorem: Memoryless nonlinearities are frequency invariant in the sense of (33). Thus, for such systems, (39) is exact, i.e.,

$$\Delta V(nf + \Delta f) = \varepsilon \frac{\partial v_n}{\partial i_m} \tag{42}$$

Thus, for memoryless nonlinearities, the off-frequency approach gives the same answer as the on-frequency method. Likewise, the approximations in (40) and (41) are replaced with equalities.

The proof is in Appendix A.

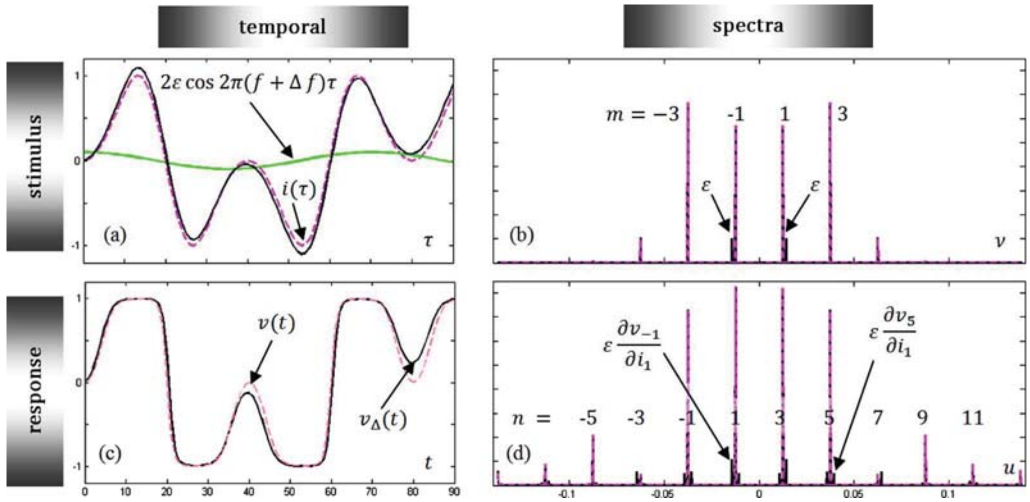

Fig. 4. A simulation illustration the off-frequency method of finding harmonic coupling weights. A period of the stimulus, $i(\tau) = \sin(\pi\cos(2\pi f\tau))$ where $f = 50/4000$, is shown in (a) with the tickle tone $2\varepsilon\cos(2\pi(f+\Delta f)\tau)$ where $\Delta f = 7/4000$ and $\varepsilon = 0.10$. The sum of the two signals is shown in (a) with the dark bold line. The magnitude of the Fourier transform (spectrum) of this signal, shown in (b), is graphically indistinguishable from the spectrum of $i(t)$ except for the tickle tones with weight appearing at $\pm(f+\Delta f)$. (The signal $i(\tau)$ has only odd harmonics.) Shown in (c) with the light dashed line is the response $v(t) = g(i(t)) = \tanh(3i(t))$. The solid line is the response to the perturbation which is $v_\Delta(t) = \tanh(3(i(t) + 2\varepsilon\cos(2\pi(f+\Delta f)t)))$. The magnitude of the spectra of $v(t)$ and $v_\Delta(t)$, shown in (d), are again graphically indistinguishable except for the small terms at $\pm\Delta f$ from each harmonic. The amplitude of these terms are mined to (for positive frequencies) to obtain the harmonic coupling weights about the operating points of $i(\tau)$ and $v(t)$.

A. Using Real Tickle Tones

When real tickle tones are used, the input has stimulus frequencies at the two harmonic numbers $\pm m$. For the on-frequency approach, the frequencies are at $\nu = \pm mf$. The output partials are estimating by examining the perturbations at $u = nf$ for $n \geq 0$. For the off-frequency method, the stimulus is applied at the two frequencies $\nu = \pm(mf + \Delta f)$ and the outputs are examined at $u = nf \pm \Delta f$ for nonnegative n.

In lieu of the two tickle tones in (17) and (18), we now use a single off-frequency tickle tone

$$\begin{aligned}\Delta_k^c i(\tau) &= 2\varepsilon\cos\left(2\pi\left(mf+\Delta f\right)\tau\right)\\ &= \varepsilon\left(e^{j2\pi(mf+\Delta f)\tau} + e^{-j2\pi(km+\Delta f)\tau}\right)\end{aligned} \tag{43}$$

Besides frequency invariance, an additional *harmonic noninterference constraint* is now required. Specifically, consider the response to two perturbations

$$v_{\pm\Delta}(t) = \mathcal{Z}\left(i(\tau) + \varepsilon e^{\pm j2\pi(mf+\Delta f)\tau}\right).$$

The constraint requires

$$V_{+\Delta}(nf - \Delta f) \approx V(nf - \Delta f)$$

and

$$V_{-\Delta}(nf + \Delta f) \approx V(nf + \Delta f).$$

In other words, the input perturbation of $\varepsilon e^{j2\pi(mf+\Delta f)\tau}$ has no effect on the response spectrum at the frequencies $u = nf - \Delta f$ and $\varepsilon e^{-j2\pi(mf+\Delta f)\tau}$ has no effect at $u = nf + \Delta f$. Then the following harmonic coupling weights can thus be read directly from the spectrum. Specifically, we read

$$\varepsilon\frac{\partial v_n}{\partial i_m} \text{ at frequency } u = nf + \Delta f$$

and

$$\varepsilon\frac{\partial v_{-n}}{\partial i_m} \text{ at frequency } u = nf - \Delta f.$$

The off-frequency technique has the advantage of determining both the harmonic coupling weights $\partial v_n/\partial i_m$ and $\partial v_{-n}/\partial i_m$ with a single experiment.

This is further illustrated in Fig. 3.

A simulation illustration using a hyperbolic tangent to model an amplifier with saturation is shown in Fig. 4. Details are in the caption.

1) Off-Frequency Method for Polynomial PP Systems: We illustrate the off-frequency method for cases where the PP operator can be described by a polynomial in (28). Such systems, we show, strictly adhere to the harmonic noninterference constraint.

We perturb the system with an mth order harmonic tickle tone $\Delta i(t)$ at a frequency offset of Δf Using a binomial expansion about an operating point $i(t)$ gives

$$\begin{aligned}v_\Delta(t) &= \sum_{p=0}^{P}\alpha_p\left(i(t) + 2\varepsilon\cos\left(2\pi\left(mf+\Delta f\right)t\right)\right)^p\\ &\approx v(t) + \Delta v(t)\end{aligned} \tag{43a}$$

We show in Appendix B that

$$\begin{aligned}\Delta v(t) = \varepsilon&\sum_{n=-\infty}^{\infty}\frac{\partial v_n}{\partial i_m}e^{j2\pi(nf+\Delta f)t}\\ &+\varepsilon\sum_{n=-\infty}^{\infty}\frac{\partial v_n}{\partial i_{-m}}e^{j2\pi(nf-\Delta f)t}\end{aligned} \tag{44}$$

Therefore, from the single off-frequency experiment using a single cosine tickle tone, both $\partial v_n/\partial i_m$ and $\partial v_n/\partial i_{-m}$ can be measured. For the on-frequency technique, two experiments

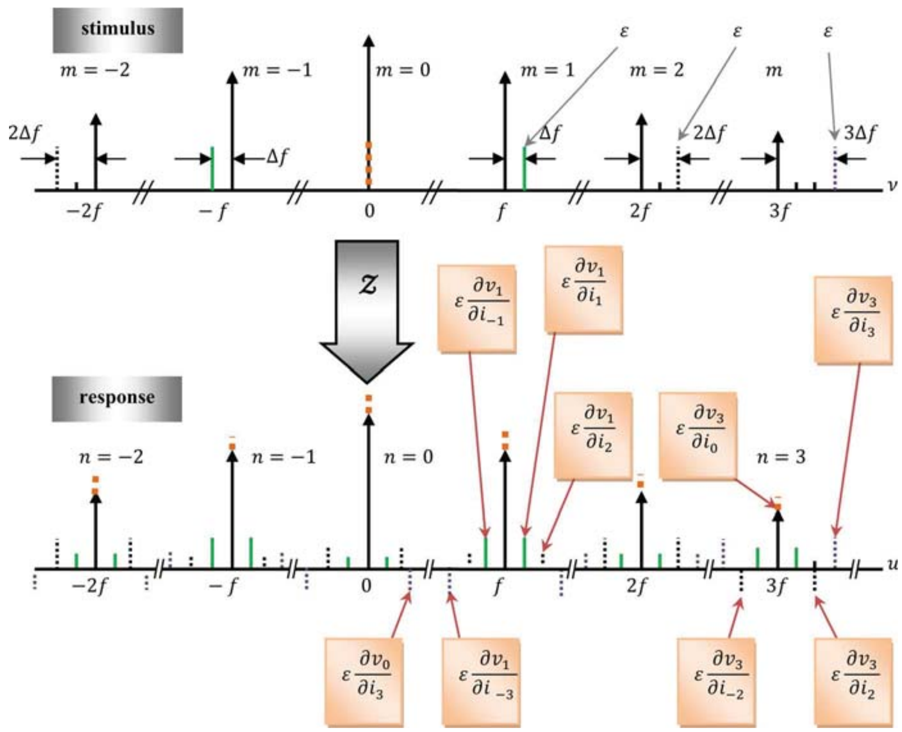

Fig. 5. Illustration of simultaneous multiplexed off-frequency measurement of numerous harmonic coupling weights. As is shown in the top figure, the stimulus, $i(t)$ has cosinusoid tickle tones applied at frequencies $m(f+\Delta f)$ there $T=1/f$ is the period of the periodic signals $i(t)$ and $v(t)$. In this example, all of the amplitudes are ε. If the PP system conforms to frequency invariant and cross interference constraints, then numerous harmonic coupling weights can be measured from the response as shown in the bottom figure. At frequencies $u=nf\pm m\Delta f$ we measure $\varepsilon(\partial v_n/\partial i_m)$. A simulation is shown in Fig. 6.

were needed: one for a cosine tickle tone and another with a sine.

V. Multiplexing Off-Frequency Measurements

Within the limitations of the model, numerous harmonic coupling weights can be measured simultaneously in a single measurement. Specifically, for some range of n and m, a single experiment suffices to measure $\partial v_n/\partial i_m$ and $\partial v_n/\partial i_{-m}$.

Let

$$i_\Delta(\tau)=i(\tau)+\Delta i(\tau)$$

where $i(t)$ has a Fourier series in (10). The perturbation, $\Delta i(\tau)$ is assumed to be a trigonometric polynomial [26] with fundamental frequency $f+\Delta f$, i.e.,

$$\Delta i(\tau)=\sum_{m=-M}^{M}\Delta i_m e^{j2\pi m(f+\Delta f)\tau} \tag{45}$$

where M is the number of harmonic coupling weights to be measured. For the special case where all of the perturbations are the same, i.e., $\Delta i_m=\varepsilon$, (45) becomes the superposition of tickle tones

$$\begin{aligned}\Delta i(\tau)&=\varepsilon\sum_{m=-M}^{M}e^{j2\pi m(f+\Delta f)\tau}\\&=(2M+1)\,\mathrm{array}_{2M+1}((f+\Delta f)\tau)\end{aligned} \tag{46}$$

where the array function is defined by [26]

$$\mathrm{array}_K(\tau):=\frac{\sin(\pi K v)}{K\sin(\pi v)}$$

As $M\to\infty$, $(2M+1)\mathrm{array}_{2M+1}(\tau)$ approaches a string of Dirac deltas [26].

Our goal is to use $i_\Delta(t)$ to determine

$$\frac{\partial v_n}{\partial i_m}\text{ and }\frac{\partial v_n}{\partial i_{-m}}\text{ for }0\le m\le M.$$

When $\Delta i(\tau)$ is real, $\Delta i_{-m}=\Delta i_m^*$. We assume all of the frequency components, as illustrated in Figs. 5 and 6, obey not only the frequency noninterference constraint, but a cross frequency noninterference constraint, e.g., the tickle tone components at, say, $m=2$, are assumed to not effect the response perturbation for $m=1$ When this constraint is satisfied and the frequency invariant property is sufficiently present, then the stimulus component at $\nu=m(f+\Delta f)$ with weight Δi_m manifests itself at the response frequencies $u=nf+m\Delta f$ with amplitude $\Delta i_m(\partial v_n/\partial i_m)$. Likewise, the $\Delta i_{-m}=\Delta i_m^*$ stimulus terms $\nu=m(f-\Delta f)$ appear in the response with weight $\Delta i_{-m}(\partial v_n/\partial i_{-m})$ at frequencies $u=nf-m\Delta f$. Thus, a single experiment suffices to measure numerous harmonic coupling weights.

A. Polynomials

The multiplexing off-frequency approach can be derived analytically for memoryless polynomials in (28) and shown to be

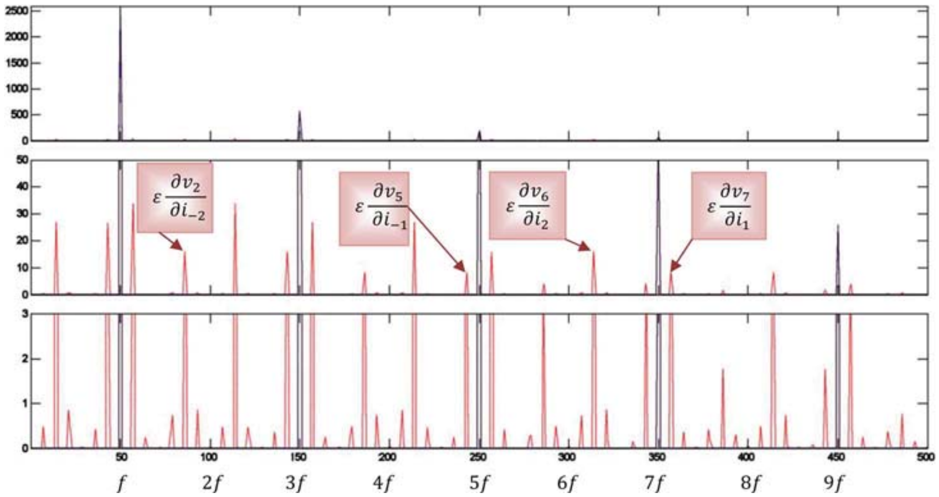

Fig. 6. A simulation demonstrating the multiplexing tickle tones illustrated in Fig. 5. The input operating point is $i(t) = \cos(2\pi f t)$ where $f = 50$. The input is subjected to a nonlinearity, $g(i) = \tanh(3i)$. Two tickle tones are added to the input equal to $\Delta i(t) = 2\varepsilon\cos(2\pi(f+\Delta f)t) + 2\varepsilon\cos(4\pi(f+\Delta f)t)$ where $\Delta f = 7$ and $2\varepsilon = 0.0250$. The signal $i_\Delta(t) = i(t) + \Delta i(t)$ for $t = 1 : 4000$ is passed through the nonlinearity to give the output, $v_\Delta(t) = g(i(t))$. The magnitude of the Fourier transform (DFT) of the output, $|V_\Delta(u)|$, is shown above. The three plots shown are of the same function, albeit over different ranges. Some harmonic coupling weights are labeled in the middle plot. Note also that the Toeplitz nature of the HCWs is illustrated in this simulation. Since $2-(-2) = 6-2$, the HCWs $\partial v_2/\partial i_{-2}$ and $\partial v_6/\partial i_2$ are the same in accordance to (24). Likewise, $\partial v_5/\partial i_{-1}$ and $\partial v_7/\partial i_1$ are the same.

exact.[2] For the polynomial in (28) we obtain the perturbed response linearity in (30). In Appendix C, we show the weight of the spectrum of $\Delta v(t)$ at frequency $nf + m\Delta f$ due the perturbation in (45) is

$$\Delta i_m \sum_{p=0}^{P} p\,\alpha_p i_{n-m}^{(p-1)^*}\Delta i_n = \frac{\partial v_n}{\partial i_m}\Delta i_m \tag{47}$$

Since Δi_m can be measured from the tickle signal, a number of harmonic coupling weights can be generated from a single measurement. For polynomial nonlinearities, the multiplexing off-frequency approach is mathematically exact.

VI. Estimation of the Hessian

The accuracy of HCW's in assessing PP systems has been perfomed on a number of test cases [33]. A overarching theoretical measure of the extent to which the HCW approximation is accurate can be estimated by deviation from a second order fit at the operating point. Alternately, a Taylor series with a second order term will be more accurate than an affine approximation. Without elaboration, we note that higher order terms can be experimentally estimated generalizing the technique used for finding harmonic coupling coefficients.

While the harmonic coupling coefficients represent the Jacobian of the $\mathcal{Z}$ operator, the second order measure is the Hessian with elements

$$\frac{\partial^2 v_n}{\partial i_\ell \partial i_m}$$

This can be estimated by placing stimulus tickle tones simultaneously at frequencies $\nu = \ell f$ and $\nu = mf$ and measuring the perturbations at response frequencies $u = nf$. Extentions to use of real tickle tones and off-frequency methods are possible.

[2] In practice, doing so is unnecessary because, for memoryless nonlinearities, a single set of measurements is needed to specify the defining function in (24).

When $\ell = m$ the second order Hessian can be estimated from the difference between first order Jacobian derivatives using tickle tones with amplitudes ε and $-\varepsilon$. Estimation of higher order derivatives becomes more susceptable to measurement uncertainty [26], [34].

VII. Conclusions

The impulse response of of an LTI system totally characterizes the system. The response to any stimuli can be found through a convolution of the impulse response with the stimulus signal. Likewise, the harmonic coupling weights (HCWs) of a PP system characterizes response to any small variation about a fixed point. The HWCs can be measured experimentally using either on-frequency or off-frequency methods.

Appendix

Proof of (42) for Memoryless Nonlinearities: Using inverse Fourier transforms, the nonlinearity in (23) can be written as

$$\begin{aligned} V(u) &= \int_{-\infty}^{\infty} g(i(t))\, e^{-j2\pi ut} dt \\ &= \int_{-\infty}^{\infty} g\left(\int_{-\infty}^{\infty} I(w) e^{j2\pi wt} dw\right) e^{-j2\pi ut} dt \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial V(u)}{\partial I(\nu)} &= \int_{-\infty}^{\infty} \dot{g}(i(t)) \left(\int_{-\infty}^{\infty} \frac{\partial I(w)}{\partial I(\nu)} e^{j2\pi wt} dw\right) e^{-j2\pi ut} dt \\ &= \int_{-\infty}^{\infty} \dot{g}(i(t)) e^{-j2\pi(u-\nu)t} dt \end{aligned}$$

Let $G_{i(t)}(u)$ be the Fourier transform of $\dot{g}(i(t))$. Then

$$\frac{\partial V(u)}{\partial I(\nu)} = G_{i(t)}(\nu - u) = \frac{\partial V(u-\alpha)}{\partial I(\nu-\alpha)}$$

for any α. Therefore, when a PP system is a memoryless nonlinearity, (39) is true with no approximations

$$\Delta V(nf+\Delta f) = \varepsilon \frac{\partial V(nf+\Delta f)}{\partial I(mf+\Delta f)} = \varepsilon \frac{\partial V(nf)}{\partial I(mf)} = \varepsilon \frac{\partial v_n}{\partial i_m}$$

Q.E.D.

Proof of (44): From (43) we apply a binomial expansion

$$v_\Delta(t) = \sum_{p=0}^{P} \alpha_p \sum_{q=0}^{p} \binom{p}{q} (2\varepsilon)^q \left(\cos\left(2\pi\left(mf + \Delta f\right)t\right)\right)^q i^{p-q}(t).$$

If $\ll 1$, we can discard terms containing ε^2 or higher. The result is the affine approximation

$$v_\Delta(t) \simeq \sum_{p=0}^{P} \alpha_p \left(i^p(t) + 2p\varepsilon i^{p-1}(t)\cos\left(2\pi\left(mf+\Delta f\right)t\right)\right) = v(t) + \Delta v(t)$$

where

$$\Delta v(t) = 2\varepsilon \sum_{p=0}^{P} p\alpha_p i^{p-1}(t)\cos\left(2\pi\left(mf+\Delta f\right)t\right)$$

Then using Euler's identity and the Fourier series expansion of $i^{p-1}(t)$,

$$\begin{aligned}\Delta v(t) &= \varepsilon \sum_{p=0}^{P} p\,\alpha_p \left[\sum_{q=-\infty}^{\infty} i_q^{(p-1)^*} e^{j2\pi qft}\right] \times \left(e^{j2\pi(mf+\Delta f)t} + e^{-j2\pi(mf+\Delta f)t}\right) \\ &= \varepsilon \sum_{p=0}^{P} p\,\alpha_p \left[\sum_{q=-\infty}^{\infty} i_q^{(p-1)^*} e^{j2\pi((q+m)f+\Delta f)t} + \sum_{q=-\infty}^{\infty} i_q^{(p-1)^*} e^{j2\pi((q-m)f-\Delta f)t}\right]\end{aligned}$$

Substitute $n = q+m$ in the first sum over q and, in the second, $n = q-m$. Interchanging sums gives

$$\Delta v(t) = \varepsilon \sum_{n=-\infty}^{\infty} \left(\sum_{p=0}^{P} p\,\alpha_p i_{n-m}^{(p-1)^*}\right) e^{j2\pi(nf+\Delta f)t} + \varepsilon \sum_{n=-\infty}^{\infty} \left(\sum_{p=0}^{P} p\,\alpha_p i_{n+m}^{(p-1)^*}\right) e^{j2\pi(nf-\Delta f)t}$$

Thus, from (24) and (32), we obtain (44).

The value of $\partial v_n/\partial i_m$ can thus be determined at by measuring the value at frequency $nf+\Delta f$. Likewise, at the frequency $nf-\Delta f$ we can measure $\partial v_n/\partial i_{-m}$.

$$\sum_{p=0}^{P} p\,\alpha_p \sum_{k=-\infty}^{\infty} i_k^{(p-1)^*} e^{j2\pi kft} \sum_{m=-M}^{M} \Delta i_m e^{j2\pi m(f+\Delta f)t}$$

Proof of (47): Substituting the off-frequency Fourier series in (45) into (30) gives

$$\begin{aligned}\Delta v(t) &= \sum_{p=0}^{P} p\,\alpha_p \sum_{k=-\infty}^{\infty} i_k^{(p-1)^*} e^{j2\pi kft} \sum_{m=-M}^{M} \Delta i_m e^{j2\pi m(f+\Delta f)t} \\ &= \sum_{p=0}^{P} p\,\alpha_p \sum_{m=-M}^{M} \Delta i_m \sum_{k=-\infty}^{\infty} i_k^{(p-1)^*} e^{j2\pi((m+k)f+m\Delta f)t}\end{aligned}$$

Substituting $n = k+m$ in the k sum gives

$$\begin{aligned}\Delta v(t) &= \sum_{p=0}^{P} p\,\alpha_p \sum_{n=-\infty}^{\infty} \left[\sum_{m=-M}^{M} i_{n-m}^{(p-1)^*} \Delta i_m e^{j2\pi m\Delta ft}\right] e^{j2\pi nft} \\ &= \sum_{m=-M}^{M} \left[\sum_{n=-\infty}^{\infty} \left(\sum_{p=0}^{P} p\,\alpha_p i_{n-m}^{(p-1)^*}\right) e^{j2\pi(nf+m\Delta f)t}\right] \Delta i_m\end{aligned}$$

The weights of this signal for exponential sinusoids at frequencies $u = nf + m\Delta f$ gives (47).

Acknowledgment

The authors would like to express appreciation to Lawrence Cohen, Jean de Graaf, and Dr. Eric Mokole of NRL for their technical expertise and advice throughout this research. The contributions of Baylor student research assistants Josh Martin and Matthew Moldovan are greatly appreciated. The authors also thank the reviewers and the Associate Editor for a very in-depth and helpful review process. The paper is of higher quality because of their efforts.

References

[1] C. Baylis, II and R. J. Marks, II, "Frequency multiplexing tickle tones to determining harmonic coupling weights in nonlinear systems," in *Proc. 78th ARFTG Microw. Measure. Conf.*, Tempe, AZ, Nov. 29–Dec. 2 2011.

[2] C. Baylis, R. J. Marks, II, J. Martin, H. Miller, and M. Moldovan, "Going nonlinear," *IEEE Microw. Mag.*, vol. 12, no. 2, pp. 55–64, Apr. 2011.

[3] C. Baylis and R. J. Marks, II, "Small perturbation harmonic coupling in nonlinear periodicity preservation circuits," *IEEE Trans. Circuits Syst. I, Reg. Papers* [Online]. Available: http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=6248714

[4] L. Betts, "Vector and harmonic amplitude/phase corrected multi-envelope stimulus response measurements of nonlinear devices," in *Proc. IEEE MTT-S Int. Microw. Symp.*, 2008, pp. 261–264.

[5] M. Isaksson *et al.*, "A comparative analysis of behavioral models for RF power amplifiers," *IEEE Trans. Microw. Theory Tech.*, vol. 54, no. 1, pp. 348–359, Jan. 2006.

[6] J. A. Jargon, K. C. Gupta, and D. C. DeGroot, "Nonlinear large-signal scattering parameters: Theory and application," in *63rd ARFTG Microw. Measure. Conf. Dig.*, Jun. 2004, pp. 157–174.

[7] J. C. Pedro *et al.*, "A comparative overview of microwave and wireless power-amplifier behavioral modeling approaches," *IEEE Trans. Microw. Theory Tech.*, vol. 53, no. 4, pp. 1150–1163, Apr. 2005.
[8] J. Wood and D. E. Root, *Fundamentals of Nonlinear Behavioral Modeling for RF and Microwave Design*. Norwood, MA: Artech House, 2005, ch. 3.
[9] J. Wood and G. Collins, "Investigation of X-parameters measurements on a 100 W Doherty power amplifier," in *Proc. 75th Microw. Measure. Conf. (ARFTG)*, 2010, pp. 1–7, DOI: 10.1109/ARFTG.2010.5496319.
[10] G. Simpson, "High power load pull with X-parameters—A new paradigm for modeling and design," in *Proc. IEEE 11th Annu. Wireless Microw. Technol. Conf. (WAMICON),* , Apr. 12–13, 2010, pp. 1–4, doi: 10.1109/WAMICON.2010.5461843.
[11] F. Verbeyst and M. V. Bossche, "VIOMAP, the S-parameter equivalent for weakly nonlinear RF and microwave devices," *IEEE Trans. Microw. Theory Tech.*, vol. 42, no. 12, pp. 2531–2535, Dec. 1994.
[12] M. Myslinski, F. Verbeyst, F. M. B. Vanden, and D. Schreurs, "A method to select correct stimuli levels for S-functions behavioral model extraction," in *IEEE MTT-S Int. Microw. Symp. Dig. (MTT)*, May 23–28, 2010, p. 1, doi: 10.1109/MWSYM.2010.5518233.
[13] G. Sun, Y. Xu, and A. Liang, "The study of nonlinear scattering functions and X-parameters," in *Proc. 2010 Int. Conf. Microw. Millimeter Wave Technol. (ICMMT)*, pp. 1086–1089, DOI:10.1109/ICMMT.2010.5525111.
[14] J. Verspecht, M. V. Bossche, and F. Verbeyst, "Characterizing components under large signal excitation: Defining sensible 'large signal S-parameters," in *49th ARFTG Conf. Dig.*, 1997, pp. 109–117.
[15] J. Verspecht, D. F. Williams, D. Schreurs, K. A. Remley, and M. D. McKinley, "Linearization of large-signal scattering functions," *IEEE Trans. Microw. Theory Tech.*, vol. 53, no. 4, pp. 1369–1376, 2005.
[16] C.-S. Chiu, K.-M. Chen, G.-W. Huang, C.-H. Hsiao, K.-H. Liao, W.-L. Chen, S.-C. Wang, M.-Y. Chen, Y.-C. Yang, K.-L. Wang, and L.-K. Wu, "Characterization of annular-structure RF LDMOS transistors using polyharmonic distortion model," in *IEEE MTT-S Int. Microwave Symp. Dig.*, 2009, pp. 977–980, MTT '09, DOI: 10.1109/MWSYM.2009.5165862.
[17] C.-S. Chiu, K.-M. Chen, G.-W. Huang, S.-Y. Lin, B.-Y. Chen, C.-C. Hung, S.-Y. Huang, C.-W. Fan, C.-Y. Tzeng, and S. Chou, "Power improvement for 65nm nMOSFET with high-tensile cesl and fast nonlinear behavior modeling," in *Proc. 2010 IEEE Radio Freq. Integr. Circuits Symp. (RFIC)*, pp. 589–592, DOI: 10.1109/RFIC.2010.5477258.
[18] C.-S. Chiu, S.-Y. Lin, B.-Y. Chen, K.-M. Chen, and G.-W. Huang, "Nonlinear behavior characterization of RF active devices using impedance-dependence X-parameters," in *Proc. Asia-Pacific Microw. Conf. (APMC),* , Dec. 7–10, 2010, pp. 2307–2310.
[19] D. E. Root, J. Verspecht, D. Sharrit, J. Wood, and A. Cognata, "Broadband polyharmonic distortion (PHD) behavioral models from fast automated simulations and large-signal vectorial network measurements," *IEEE Trans. Microw. Theory Tech.*, vol. 53, no. 11, pp. 3656–3664, Nov. 2005.
[20] R. S. Saini, S. Woodington, J. Lees, J. Benedikt, and P. J. Tasker, "An intelligence driven active loadpull system," in *Proc. 2010 75th Microw. Measure. Conf. (ARFTG)*, pp. 1–4, DOI: 10.1109/ARFTG.2010.5496327.
[21] J. Verspecht *et al.*, "Multi-port, and dynamic memory enhancements to PHD nonlinear behavioral models from large-signal measurements and simulations," in *Conf. Rec. IEEE Microw. Theory Tech. Symp.*, Jun. 2007, pp. 969–972.
[22] J. Verspecht, J. Horn, and D. E. Root, "A simplified extension of X-parameters to describe memory effects for wideband modulated signals," in *Proc. 75th ARFTG Microw. Measure. Conf. (ARFTG)*, May 28–28, 2010, pp. 1–6, doi: 10.1109/ARFTG.2010.5496334.
[23] J. Verspecht, J. Horn, L. Betts, D. Gunyan, R. Pollard, C. Gillease, and D. E. Root, "Extension of X-parameters to include long-term dynamic memory effects," in *Proc. 2009 IEEE Int. Microw. Symp.*, Boston, MA.
[24] D. T. Bespalko and S. Boumaiza, "X-parameter measurement challenges for unmatched device characterization," in *Proc. 2010 75th Microw. Measure. Conf. (ARFTG)*, pp. 1–4, DOI: 10.1109/ARFTG.2010.5496317.
[25] C.-S. Chiu, S.-Y. Lin, B.-Y. Chen, K.-M. Chen, and G.-W. Huang, "Nonlinear behavior characterization of RF active devices using impedance-dependence X-parameters," in *Proc. Asia-Pacific Microw. Conf. (APMC),* , Dec. 7–10, 2010, pp. 2307–2310.
[26] R. J. Marks, II, *Handbook of Fourier Analysis and Its Applications*. New York: Oxford Univ. Press, 2009.
[27] R. J. Marks, II, J. F. Walkup, and M. O. Hagler, "A sampling theorem for space-variant systems," *J. Opt. Soc. Amer.*, vol. 66, no. 9, pp. 918–921, 1976.
[28] W. T. Rhodes and J. Florence, "Frequency variant optical signal analysis," *Appl. Opt.*, vol. 15, no. 12, pp. 3073–3073, 1976.
[29] R. J. Marks, II, J. F. Walkup, M. O. Hagler, and T. F. Krile, "Space-variant processing of one-dimensional signals," *Appl. Opt.*, vol. 16, pp. 739–745, 1977.
[30] T. F. Krile, R. J. Marks, II, J. F. Walkup, and M. O. Hagler, "Holographic representations of space-variant systems using phase-coded reference beams," *Appl. Opt.*, vol. 16, no. 12, pp. 3131–3135, 1977.
[31] R. J. Marks, II, J. F. Walkup, and M. O. Hagler, "Sampling theorems for linear shift-variant systems," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 228–233, 1978.
[32] B. Hicks, L. Braida, and N. Durlach, "Pitch invariant frequency lowering with nonuniform spectral compression," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Apr. 1981, vol. 6, pp. 121–124, doi: 10.1109/ICASSP.1981.1171344.
[33] J. Martin, C. Baylis, R. J. Marks, II, and M. Moldovan, "Perturbation size and harmonic limitations in affine approximation for time invariant periodicity preservation systems," in *Proc. Int. Waveform Diversity Design Conf.*, Jan. 2012, pp. 22–27.
[34] R. J. Marks, II and M. W. Hall, "Differintegral interpolation from a bandlimited signal's samples," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-29, pp. 872–877, 1981.

Charles Baylis, II (S'03–M'08) received the D.Phil. in electrical engineering, University of South Florida, Tampa, in 2007.

He is Assistant Professor and the Co-Director of the Wireless and Microwave Circuits and Systems Program (WMCSLab.org) in the Department of Electrical and Computer Engineering, Baylor University, Waco, TX. For 2007–2008 he was a Visiting Assistant Professor at Department of Electrical Engineering, University of South Florida. He specializes in high-efficiency power amplifier design for radar and communication systems, nonlinear microwave measurements, nonlinear transistor modeling techniques and approaches, related system applications.

Dr. Baylis was the General Chair of the First and Second Mini-Symposia on Wireless and Microwave Circuits and Systems, Waco, April 2009 and March 2010. He serves on the executive Committee Member and Student Activities Chair of the IEEE Microwave Theory and Techniques Society, Dallas Chapter, 2009–2011.

Robert J. Marks, II (S'71–M'72–SM'83–F'94) is Distinguished Professor of Electrical and Computer Engineering, Baylor University, Waco, TX. His consulting activities include Microsoft Corporation, Pacific Gas & Electric, and Boeing Computer Services. His research has been funded by organizations such as the National Science Foundation, General Electric, Southern California Edison, EPRI, the Air Force Office of Scientific Research, the Office of Naval Research, the Whitaker Foundation, Boeing Defense, the National Institutes of Health, The Jet Propulsion Lab, Army Research Office, and NASA.

Prof. Marks is a Fellow of the Optical Society of America. He is the recipient of numerous professional awards, including a NASA Tech Brief Award and a best paper award from the American Brachytherapy Society for prostate cancer research. He was awarded the IEEE Outstanding Branch Councilor Award, The IEEE Centennial Medal, the IEEE Neural Networks Society Meritorious Service Award, the IEEE Circuits and Systems Society Golden Jubilee Award and, for 2007, the IEEE CIS Chapter of the IEEE Dallas Section Volunteer of the Year award. He was named a Distinguished Young Alumnus of Rose-Hulman Institute of Technology, is an inductee into the Texas Tech Electrical Engineering Academy, and in 2007 was awarded the Banned Item of the Year from the Discovery Institute. He served for 17 years as the faculty advisor to the University of Washington's chapter of Campus Crusade for Christ. In 2010, he was listed by CollegeCrunch.com as one of "The 20 Most Brilliant Christian Professors."